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# Gauge and Lorentz covariant quark propagator in an arbitrary gluon field

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**Abstract.** The quark propagator in the presence of an arbitrary gluon field is calculated gauge and Lorentz covariantly order by order in terms of powers of the gluon field and its derivatives. The result is independent of the path connecting the ends of the propagator, and the leading order result coincides with the exact propagator in the trivial case of a vanishing gluon field.

The quark propagator $S(x,y;A) = (\mathrm{i}\nabla\!\!\!/ - m)^{-1}(x,y)$[1] with $\nabla^\mu = \partial^\mu - \mathrm{i}gA^\mu$ for a quark of mass $m$ in the presence of a gluon field $A^\mu$ plays an important role in many investigations of quantum chromodynamics. Integrating out the gluon field, we get the physical quark propagator:

$$\begin{aligned} &-\mathrm{i}\langle 0|\mathbf{T}\psi(x)\overline{\psi}(y)|0\rangle \\ &= \frac{\int \mathcal{D}A_\mu \; S(x,y;A)\mathrm{e}^{\mathrm{i}S_{\mathrm{QCD}}(A)}}{\int \mathcal{D}A_\mu \; \mathrm{e}^{\mathrm{i}S_{\mathrm{QCD}}(A)}} \equiv \langle S(x,y;A)\rangle \,, \end{aligned} \quad (1)$$

where $S_{\mathrm{QCD}}(A)$ is the QCD effective action with the quark field integrated out. The propagator $S(x,y;A)$ is Lorentz covariant and under a color gauge transformation $A_\mu(x) \to V(x)A_\mu(x)V^\dagger(x) - \mathrm{i}/gV(x)[\partial^\mu V^\dagger(x)]$ it transforms as

$$S(x,y;A) \to S'(x,y;A) = V(x)S(x,y;A)V^\dagger(y) \,. \quad (2)$$

Except for the formal expression of $S(x,y;A)$, an expansion for $S(x,y;A)$ in terms of the local gluon field $A$ is expected and plays a key role in its applications. Note that the bilocal transformation law (2) prohibits the naive expansion $\sum_n C_n(x-y)O_n[A(x)]$ for $S(x,y;A)$ with $C_n(x-y)$ being a gluon field independent coefficient and $O_n[A(x)]$ a local operator depending on the gluon field $A_\mu(x)$, since it is impossible for a local operator $O_n[A(x)]$ to transform bilocally. We will set up a modified expansion by multiplying the naive expansion by a local $A_\mu(x)$ dependent "phase factor" $\mathbf{a}[x-y;A(x)]$ with bilocal transformation law $\mathbf{a}[x-y;A(x)] \to \mathbf{a}'[x-y;A(x)] = V(x)\mathbf{a}[x-y;A(x)]V^\dagger(y)$,

$$S(x,y;A) = \left[\sum_n C_n(x-y)O_n[A(x)]\right] \mathbf{a}[x-y;A(x)] \,, \quad (3)$$

which will match the transformation law (2). Then (1) implies that we can expand the physical quark propagator as follows:

$$\begin{aligned} &-\mathrm{i}\langle 0|\mathbf{T}\psi(x)\overline{\psi}(y)|0\rangle = \langle S(x,y;A)\rangle \\ &= \sum_n C_n(x-y)\langle O_n[A(x)] \; \mathbf{a}[x-y,A(x)]\rangle \,, \end{aligned} \quad (4)$$

which can be treated as a gauge covariant modified operator product expansion for the quark propagator.

In the literature, the most simple approximation for the expansion of $S(x,y;A)$ is based on the perturbation expansion

$$\begin{aligned} &S(x,y;A) \\ &= \left[[1 + (\mathrm{i}\partial\!\!\!/ - m)^{-1}gA\!\!\!/\,]^{-1}(\mathrm{i}\partial\!\!\!/ - m)^{-1}\right](x,y) \\ &= (\mathrm{i}\partial\!\!\!/ - m)^{-1}(x,y) \\ &- \left[(\mathrm{i}\partial\!\!\!/ - m)^{-1}gA\!\!\!/\,(\mathrm{i}\partial\!\!\!/ - m)^{-1}\right](x,y) \\ &+ \left[(\mathrm{i}\partial\!\!\!/ - m)^{-1}gA\!\!\!/\,(\mathrm{i}\partial\!\!\!/ - m)^{-1}gA\!\!\!/\,(\mathrm{i}\partial\!\!\!/ - m)^{-1}\right](x,y) \\ &+ \cdots \,, \end{aligned} \quad (5)$$

where the expansion can be calculated up to arbitrary orders. The result can be directly expressed in terms of powers of the gluon field and its differentials, but to any fixed order of the calculation, gauge covariance is violated. Another approximation is the so called static approximation proposed by Brown and Weisberger [1] and Eichten and Feinberg [2]. By neglecting the spatial part of $\nabla\!\!\!/$, it leads to

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[1] Our formulae are given in Minkowski space with Bjorken–Drell conventions.

$$
\begin{aligned}
&S_{\text{static}}(x,y;A)\\
&= -\frac{\mathrm{i}}{2}[\theta(x_0-y_0)(1+\gamma^0)+\theta(y_0-x_0)(1-\gamma^0)]\\
&\times\delta(\mathbf{x}-\mathbf{y})\mathrm{e}^{-im|x_0-y_0|}\mathbf{P}\mathrm{e}^{\mathrm{i}g\int_y^x \mathrm{d}z^0\ A_0(z)}\ , \qquad (6)
\end{aligned}
$$

where the path in the integral is the straight line from $y$ to $x$, and the path ordering is $A_0(x)$ to the left, $\cdots$, $A_0(y)$ to the right. The neglected spatial term can subsequently be taken into account as a perturbation [3]. This formalism keeps the gauge covariance of the propagator, but Lorentz covariance is lost; it even does not coincide with the exact propagator in the trivial case of a vanishing gluon field. Recently, Gromes reinvestigated the problem [4]. He, in terms of path ordered exponentials, wrote the first order perturbation theory as a non-perturbative expression which has the correct behavior under Lorentz and gauge transformations. His result is only at the lowest order; the existence of path ordered exponentials makes the expression very complex and causes the problem of path dependence. It is the purpose of the present work to invent another path independent calculation formalism which can keep the advantages of different formalisms mentioned above:

(1) The expansion can be calculated up to arbitrary orders.
(2) The result can be directly expressed in terms of powers of the gluon field and its differentials and it coincides with the exact propagator in the trivial case of a vanishing gluon field.
(3) The result is gauge and Lorentz transformation covariant.
(4) There is no path dependence of the result.

We start by writing the quark propagator as follows:

$$
S(x,y;A) = \langle x|(\mathrm{i}\not\nabla + m)(E-\nabla^2-m^2)^{-1}|y\rangle\ , \qquad (7)
$$

where $E-\nabla^2-m^2 \equiv (\mathrm{i}\not\nabla - m)(\mathrm{i}\not\nabla + m) = (\mathrm{i}\not\nabla + m)(\mathrm{i}\not\nabla - m)$ with $E = \frac{\mathrm{i}g}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}$ and $F_{\mu\nu} \equiv \frac{\mathrm{i}}{g}[\nabla_\mu,\nabla_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g[A_\mu, A_\nu]$.

The next step is to calculate the matrix element $\langle x|(\mathrm{i}\not\nabla + m)(E-\nabla^2-m^2)^{-1}|y\rangle$ which in momentum space can be written as

$$
\begin{aligned}
&\langle x|(\mathrm{i}\not\nabla + m)(E-\nabla^2-m^2)^{-1}|y\rangle\\
&= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}k\cdot z}(\mathrm{i}\not\nabla_x + \not k + m)\\
&\times[(k+\mathrm{i}\nabla_x)^2 + E(x) - m^2]^{-1}1\Big|_{z=x-y}\ . \qquad (8)
\end{aligned}
$$

Conventionally, to directly perform a Taylor expansion over the operator $\nabla_x^\mu$ and $E(x)$ will lead to the result. The gauge covariance in this calculation program is not obvious, since the operator $\nabla_x^\mu$, once acting on the final unity 1, gives $-\mathrm{i}gA^\mu(x)$, which is not a gauge covariant quantity. Only if $\nabla_x^\mu$ and $E(x)$ are composed of commutators such as $[\nabla_x^\mu, \nabla_x^\nu]$ and $[\nabla_x^\mu, E(x)]$, the gauge covariance is explicit realized. We need a formalism to explicitly exhibit this gauge covariance.

Consider

$$
\begin{aligned}
&\mathrm{e}^{\mathrm{i}\nabla_x\cdot\frac{\partial}{\partial k}}k^\mu \mathrm{e}^{-\mathrm{i}\nabla_x\cdot\frac{\partial}{\partial k}} \qquad (9)\\
&= k^\mu + \mathrm{i}\nabla_x^\mu + F\left[\left(\mathrm{i}\nabla_x\cdot\frac{\partial}{\partial k}\right)\mathrm{d}\left(\mathrm{i}\nabla_x\cdot\frac{\partial}{\partial k}\right)\right](\mathrm{i}\nabla_x^\mu)\ ,
\end{aligned}
$$

where we have used relation

$$
\begin{aligned}
\mathrm{e}^A B \mathrm{e}^{-A} &= [\mathrm{e}^{\mathrm{Ad}A}](B) = B + [A,B] + \frac{1}{2!}[A,[A,B]]\\
&+\frac{1}{3!}[A,[A,[A,B]]] + \cdots\\
&= B + [A,B] + F[\mathrm{Ad}A]([A,B])\ ,
\end{aligned}
$$

with the function $F$ and the operation $(A\mathrm{d}A)^n$, introduced in [5], defined by

$$
F(z) \equiv \frac{\mathrm{e}^z-1}{z} - 1 = \sum_{n=2}^{\infty}\frac{z^{n-1}}{n!}\ ,
$$

$$
\begin{aligned}
&(A\mathrm{d}A)^0(B) \equiv B\ ,\\
&(A\mathrm{d}A)^m(B) = [A,[A,\cdots,[A,B]],\cdots] \quad m \text{ times}\ .
\end{aligned}
$$

Equation (9) can be written

$$
k^\mu + \mathrm{i}\nabla_x^\mu = \mathrm{e}^{\mathrm{i}\nabla_x\cdot\frac{\partial}{\partial k}}\left[k^\mu + \tilde{F}^\mu\left(\nabla_x, \frac{\partial}{\partial k}\right)\right]\mathrm{e}^{-\mathrm{i}\nabla_x\cdot\frac{\partial}{\partial k}}\ , \qquad (10)
$$

in which $\tilde{F}^\mu\left(\nabla_x, \frac{\partial}{\partial k}\right)$ depends on $\nabla_x^\mu$ and $\frac{\partial}{\partial k}$;

$$
\begin{aligned}
&\tilde{F}^\mu\left(\nabla_x, \frac{\partial}{\partial k}\right) \qquad (11)\\
&\equiv -\mathrm{e}^{-\mathrm{i}\nabla_x\cdot\frac{\partial}{\partial k}}F\left[\left(\mathrm{i}\nabla_x\cdot\frac{\partial}{\partial k}\right)\mathrm{d}\left(\mathrm{i}\nabla_x\cdot\frac{\partial}{\partial k}\right)\right](\mathrm{i}\nabla_x^\mu)\mathrm{e}^{\mathrm{i}\nabla_x\cdot\frac{\partial}{\partial k}}\\
&= \frac{1}{2}[\nabla_x^\nu,\nabla_x^\mu]\frac{\partial}{\partial k^\nu} - \frac{\mathrm{i}}{3}[\nabla_x^\lambda,[\nabla_x^\nu,\nabla_x^\mu]]\frac{\partial^2}{\partial k^\lambda\partial k^\nu} + O(p^4)\ .
\end{aligned}
$$

We find that all terms in (11) are gauge covariant. For convenience of the expansion we assign to each $\nabla_x^\mu$ a momentum order $p$; then the terms $O(p^4)$ in (11) are those commutators with at least four $\nabla_x^\mu$ derivatives.

In terms of the $\tilde{F}$ function, (10) tells us that the term $k^\mu + \mathrm{i}\nabla_x^\mu$ can be expressed in terms of the $\frac{\partial}{\partial k}$ dependent but gauge covariant quantity $\tilde{F}$ multiplied by some $\frac{\partial}{\partial k}$ dependent exponential "phase factors". Applying this to (8), we have

$$
\begin{aligned}
&\langle x|(\mathrm{i}\not\nabla + m)(E-\nabla^2-m^2)^{-1}|y\rangle\\
&= \int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}k\cdot z}\mathrm{e}^{\mathrm{i}\nabla_x\cdot\frac{\partial}{\partial k}}(\tilde{\not F}_{\partial k} + \not k + m)\mathrm{e}^{-\mathrm{i}\nabla_x\cdot\frac{\partial}{\partial k}}\\
&\times\left[\mathrm{e}^{\mathrm{i}\nabla_x\cdot\frac{\partial}{\partial k}}(k+\tilde{F}_{\partial k})^2\mathrm{e}^{-\mathrm{i}\nabla_x\cdot\frac{\partial}{\partial k}} + E(x) - m^2\right]^{-1}1\Big|_{z=x-y}\\
&= \int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{\mathrm{i}\nabla_x\cdot\frac{\partial}{\partial k}}(\tilde{\not F}_{\partial k} + \not k + m)\\
&\times[(k+\tilde{F}_{\partial k})^2 + \tilde{E}_{\partial k}(x) - m^2]^{-1}\mathrm{e}^{-\mathrm{i}\nabla_x\cdot\frac{\partial}{\partial k}}\mathrm{e}^{-\mathrm{i}k\cdot z}\Big|_{z=x-y}\\
&= \int\frac{\mathrm{d}^4k}{(2\pi)^4}(\tilde{\not F}_{\partial k} + \not k + m) \qquad (12)\\
&\times[(k+\tilde{F}_{\partial k})^2 + \tilde{E}_{\partial k}(x) - m^2]^{-1}\mathrm{e}^{-\mathrm{i}k\cdot z}\mathrm{e}^{-z\cdot\nabla_x}\Big|_{z=x-y}\ ,
\end{aligned}
$$

where the $\nabla_x$ commute with $z$. In the last equality, we have dropped the total momentum space derivative terms

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4}\left[\mathrm{e}^{\mathrm{i}\nabla_x\cdot\frac{\partial}{\partial k}}-1\right] \times\Big[(\tilde{\not{F}}_{\partial k}+\not{k}+m)[(k+\tilde{F}_{\partial k})^2+\tilde{E}_{\partial k}(x)-m^2]^{-1} \times\mathrm{e}^{-\mathrm{i}k\cdot z}\mathrm{e}^{-z\cdot\nabla_x}\Big]_{z=x-y};$$

$\tilde{F}^\mu_{\partial k}\equiv\tilde{F}^\mu\left(\nabla_x,\frac{\partial}{\partial k}\right)$ and $\tilde{E}$ are all gauge transformation covariant quantities with the $\frac{\partial}{\partial k}$ dependent $\tilde{E}$ being defined by

$$\begin{aligned}\tilde{E}_{\partial k}&\equiv\tilde{E}\left(\nabla_x,\frac{\partial}{\partial k}\right)=\mathrm{e}^{-\mathrm{i}\nabla_x\cdot\frac{\partial}{\partial k}}E(x)\mathrm{e}^{\mathrm{i}\nabla_x\cdot\frac{\partial}{\partial k}}\\&=E(x)-\mathrm{i}[\nabla^\mu_x,E(x)]\frac{\partial}{\partial k^\mu}\\&\quad-\frac{1}{2}[\nabla^\nu_x,[\nabla^\mu_x,E(x)]]\frac{\partial^2}{\partial k^\nu\partial k^\mu}+\cdots\,.\end{aligned}\tag{13}$$

Equation (12) implies

$$S(x,y;A)=\tilde{S}[x-y;A(x)]\,\mathbf{a}[x-y;A(x)]\,,\tag{14}$$

with $\tilde{S}[x-y;A(x)]$ being defined by

$$\tilde{S}[z;A(x)]\equiv\int\frac{\mathrm{d}^4k}{(2\pi)^4}[\tilde{\not{F}}_{\partial k}(x)+\not{k}+m] \times[(k+\tilde{F}_{\partial k}(x))^2+\tilde{E}_{\partial k}(x)-m^2]^{-1}\mathrm{e}^{-\mathrm{i}k\cdot z}\,,\tag{15}$$

$$\mathbf{a}[z;A(x)]\equiv\mathrm{e}^{-z\cdot\nabla_x}1\,.\tag{16}$$

So the quark propagator in the presence of a gluon field consists of two parts: one is $\tilde{S}[x-y;A(x)]$ which can be seen as a generalized Fourier transformation of the momentum space quark propagator in the presence of a local gluon field; the other is a bilocal exponential "phase factor" $\mathbf{a}[x-y;A(x)]$. We now discuss these separately in detail.

For $\tilde{S}[x-y;A(x)]$, note that under the gauge transformation $V(x)$, $\nabla^\mu_x$ transforms as $\nabla^\mu_x\to V(x)\nabla^\mu_xV^\dagger(x)$ which leads to $\tilde{F}^\mu\left(\nabla_x,\frac{\partial}{\partial k}\right)\to V(x)\tilde{F}^\mu\left(\nabla_x,\frac{\partial}{\partial k}\right)V^\dagger(x)$ and $\tilde{E}\left(\nabla_x,\frac{\partial}{\partial k}\right)\to V(x)\tilde{E}\left(\nabla_x,\frac{\partial}{\partial k}\right)V^\dagger(x)$. Equation (15) then tells us that $\tilde{S}[z;A(x)]$ obeys the transformation rule

$$\tilde{S}[z;A(x)]\to V(x)\tilde{S}[z;A(x)]V^\dagger(x)\,.\tag{17}$$

To calculate $\tilde{S}[x-y;A(x)]$, we can first expand its integrand in terms of powers of commutators of $\nabla^\mu_x$, and denote by $I_n\left(k,\frac{\partial}{\partial k};A\right)$ the $n$th order of it, i.e.

$$(\tilde{\not{F}}_{\partial k}+\not{k}+m)[(k+\tilde{F}_{\partial k})^2+\tilde{E}_{\partial k}(x)-m^2]^{-1}=\sum_{n=0}I_n\left(k,\frac{\partial}{\partial k};A\right)\,,\tag{18}$$

with the convention that $\frac{\partial}{\partial k_\nu}$ is always at the r.h.s. of $k_\nu$. It is easy to find

$$I_0\left(k,\frac{\partial}{\partial k};A\right)=\frac{\not{k}+m}{k^2-m^2}=\frac{1}{\not{k}-m}\,,\tag{19}$$

which is just the free quark propagator in momentum space. Further with the help of (11) and (13), we find $I_1\left(k,\frac{\partial}{\partial k};A\right)=0$ and

$$\begin{aligned}&I_2\left(k,\frac{\partial}{\partial k};A\right)\\&=\left\{\frac{\mathrm{i}}{2}k_\sigma\gamma_\rho\gamma_5\epsilon^{\rho\sigma\mu\nu}-\frac{m}{4}[\gamma^\mu,\gamma^\nu]\right\}\frac{\mathrm{i}gF_{\mu\nu}}{(k^2-m^2)^2}\\&+\frac{\frac{\mathrm{i}g}{2}\gamma^\mu F_{\mu\nu}}{k^2-m^2}\frac{\partial}{\partial k_\nu}-\mathrm{i}g\frac{(\not{k}+m)F_{\mu\nu}k^\mu}{(k^2-m^2)^2}\frac{\partial}{\partial k_\nu}\,,\end{aligned}\tag{20}$$

$$\begin{aligned}&I_3\left(k,\frac{\partial}{\partial k};A\right)\\&=\frac{g}{3}\gamma^\mu[\nabla^\lambda_x,F_{\mu\nu}]\left\{\frac{-2g^{\lambda\nu}}{(k^2-m^2)^2}+\frac{8k_\nu k_\lambda}{(k^2-m^2)^3}\right.\\&-\frac{2k_\nu}{(k^2-m^2)^2}\frac{\partial}{\partial k^\lambda}-\frac{2k_\lambda}{(k^2-m^2)^2}\frac{\partial}{\partial k^\nu}\\&\left.+\frac{1}{k^2-m^2}\frac{\partial^2}{\partial k^\lambda k^\nu}\right\}-\frac{g}{3}(\not{k}+m)[\nabla^\lambda_x,F_{\mu\nu}]\\&\times\left\{\frac{1}{(k^2-m^2)^3}\left[-2g^{\lambda\mu}k^\nu-4g^{\lambda\nu}k^\mu-4k^\mu k^\lambda\frac{\partial}{\partial k^\nu}\right]\right.\\&\left.+\frac{1}{(k^2-m^2)^2}\left[g^{\lambda\mu}\frac{\partial}{\partial k^\nu}+2k^\mu\frac{\partial^2}{\partial k^\lambda k^\nu}\right]\right\}\\&+\frac{g}{2}\frac{(\not{k}+m)k_\lambda}{(k^2-m^2)^3}[\nabla^\lambda_x,[\gamma^\mu,\gamma^\nu]F_{\mu\nu}]\\&-\frac{g}{4}\frac{(\not{k}+m)}{(k^2-m^2)^2}[\nabla^\lambda_x,[\gamma^\mu,\gamma^\nu]F_{\mu\nu}]\frac{\partial}{\partial k_\lambda}\end{aligned}\tag{21}$$

$$\begin{aligned}&I_4\left(k,\frac{\partial}{\partial k};A\right)\\&=-\frac{\mathrm{i}g}{8}[\nabla^\rho_x,[\nabla^\lambda_x,F_{\mu\nu}]]\\&\times\left\{\frac{\gamma^\mu}{k^2-m^2}\frac{\partial^3}{\partial k^\rho\partial k^\lambda\partial k^\nu}+\frac{1}{(k^2-m^2)^2}\right.\\&\times\left[-2\gamma^\mu\left(g^{\lambda\nu}\frac{\partial}{\partial k^\rho}+g^{\nu\rho}\frac{\partial}{\partial k^\lambda}+g^{\rho\lambda}\frac{\partial}{\partial k^\nu}+k^\rho\frac{\partial^2}{\partial k^\lambda\partial k^\nu}\right.\right.\\&\left.+k^\nu\frac{\partial^2}{\partial k^\rho\partial k^\lambda}+k^\lambda\frac{\partial^2}{\partial k^\rho\partial k^\nu}\right)-(\not{k}+m)\\&\times\left(2k^\mu\frac{\partial^3}{\partial k^\rho\partial k^\lambda\partial k^\nu}+g^{\lambda\mu}\frac{\partial^2}{\partial k^\rho\partial k^\nu}+g^{\rho\mu}\frac{\partial^2}{\partial k^\lambda\partial k^\nu}\right.\\&\left.\left.+[\gamma^\mu,\gamma^\nu]\frac{\partial^2}{\partial k^\rho\partial k^\lambda}\right)\right]+\frac{1}{(k^2-m^2)^3}\Bigg(2(\not{k}+m)\\&\times\left[g^{\rho\mu}g^{\lambda\nu}+g^{\rho\mu}k^\nu\frac{\partial}{\partial k^\lambda}+g^{\rho\mu}k^\lambda\frac{\partial}{\partial k^\nu}+2k^\mu k^\rho\frac{\partial^2}{\partial k^\lambda\partial k^\nu}\right.\\&\left.+2k^\mu k^\lambda\frac{\partial^2}{\partial k^\rho\partial k^\nu}+[\gamma^\mu,\gamma^\nu]\left(g^{\rho\lambda}+k^\lambda\frac{\partial}{\partial k^\rho}+k^\rho\frac{\partial}{\partial k^\lambda}\right)\right]\\&+8\gamma^\mu\Bigg(g^{\rho\nu}k^\lambda+g^{\rho\lambda}k^\nu+g^{\nu\lambda}k^\rho\\&+k^\nu k^\lambda\frac{\partial}{\partial k^\rho}+k^\nu k^\rho\frac{\partial}{\partial k^\lambda}+k^\rho k^\lambda\frac{\partial}{\partial k^\nu}\Bigg)\end{aligned}$$

$$+ 2(\not k + m)\Big(2g^{\rho\nu}k^{\mu}\frac{\partial}{\partial k^{\lambda}} + 2g^{\rho\lambda}k^{\mu}\frac{\partial}{\partial k^{\nu}} + 2g^{\nu\lambda}k^{\mu}\frac{\partial}{\partial k^{\rho}}$$
$$+ g^{\lambda\mu}g^{\rho\nu} + g^{\lambda\mu}k^{\nu}\frac{\partial}{\partial k^{\rho}} + g^{\lambda\mu}k^{\rho}\frac{\partial}{\partial k^{\nu}}\Big)\Big)$$
$$+ \frac{1}{(k^2-m^2)^4}\Big( - 48\gamma^{\mu}k^{\rho}k^{\nu}k^{\lambda}$$
$$- 8(\not k + m)\Big(2g^{\rho\nu}k^{\mu}k^{\lambda} + 2g^{\lambda\nu}k^{\mu}k^{\rho} + g^{\lambda\mu}k^{\nu}k^{\rho}$$
$$+ g^{\rho\mu}k^{\nu}k^{\lambda} + 2k^{\lambda}k^{\rho}k^{\mu}\frac{\partial}{\partial k^{\nu}} + [\gamma^{\mu},\gamma^{\nu}]k^{\rho}k^{\lambda}\Big)\Big)\Big\}$$
$$- g^2F_{\lambda\rho}F_{\mu\nu}\Big\{ - \frac{1}{(k^2-m^2)^2}$$
$$\times\Big(\frac{1}{2}\gamma^{\lambda}g^{\rho\mu}\frac{\partial}{\partial k^{\nu}} + \frac{1}{2}\gamma^{\lambda}k^{\mu}\frac{\partial^2}{\partial k^{\rho}\partial k^{\nu}} + \frac{1}{8}\gamma^{\lambda}[\gamma^{\mu},\gamma^{\nu}]\frac{\partial}{\partial k^{\rho}}$$
$$+ \frac{1}{4}(\not k + m)g^{\lambda\mu}\frac{\partial^2}{\partial k^{\rho}\partial k^{\nu}}\Big) + \frac{1}{(k^2-m^2)^3}.$$
$$\times\Big[\gamma^{\lambda}\Big(g^{\rho\mu}k^{\nu} + g^{\rho\nu}k^{\mu} + 2k^{\mu}k^{\rho}\frac{\partial}{\partial k^{\nu}} + \frac{1}{2}k^{\rho}[\gamma^{\mu},\gamma^{\nu}]\Big)$$
$$+ \frac{1}{2}g^{\lambda\mu}(\not k + m)\Big(g^{\rho\nu} + k^{\nu}\frac{\partial}{\partial k^{\rho}} + k^{\rho}\frac{\partial}{\partial k^{\nu}}\Big)$$
$$+ k^{\lambda}(\not k + m)\Big(g^{\rho\mu}\frac{\partial}{\partial k^{\nu}} + k^{\mu}\frac{\partial^2}{\partial k^{\rho}\partial k^{\nu}}\Big)$$
$$+ \frac{(\not k + m)}{4}\Big(k^{\lambda}[\gamma^{\mu},\gamma^{\nu}]\frac{\partial}{\partial k^{\rho}} + k^{\mu}[\gamma^{\lambda},\gamma^{\rho}]\frac{\partial}{\partial k^{\nu}}$$
$$+ \frac{1}{4}[\gamma^{\lambda},\gamma^{\rho}][\gamma^{\mu},\gamma^{\nu}]\Big)\Big] - \frac{2}{(k^2-m^2)^4}(\not k + m)$$
$$\times(g^{\lambda\mu}k^{\nu}k^{\rho} + g^{\rho\mu}k^{\nu}k^{\lambda} + g^{\rho\nu}k^{\mu}k^{\lambda})\Big\}. \quad (22)$$

Correspondingly we can write

$$\tilde{S}[z;A(x)] = \sum_{n=0}\tilde{S}_n[z;A(x)]\,, \quad (23)$$

$$\tilde{S}_n[z;A(x)] \equiv \int\frac{\mathrm{d}^4k}{(2\pi)^4}I_n\left(k,\frac{\partial}{\partial k};A(x)\right)\mathrm{e}^{-\mathrm{i}k\cdot z}$$
$$= \int\frac{\mathrm{d}^4k}{(2\pi)^4}I_n(k,-\mathrm{i}z;A(x))\mathrm{e}^{-\mathrm{i}k\cdot z}\,. \quad (24)$$

After momentum integration, (23) and (24) will lead to the expansion $\tilde{S}[z;A(x)] = \sum_n C_n(z)O_n[A(x)]$ mentioned previously in (3). The leading term $\tilde{S}_0[z;A(x)] = C_0(z)$ with $O_0[A(x)] = 1$ is just the exact propagator in the trivial case of a vanishing gluon field,

$$\tilde{S}_0[z;A(x)] = \int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}k\cdot z}}{\not k - m}\,. \quad (25)$$

Now, we come to a discussion of the exponential "phase factor". It satisfies the constraints

$$\mathbf{a}[0;A(x)] = 1\,, \qquad \mathbf{a}[x-y;0] = 1\,,$$
$$(x-y)\cdot\nabla_x\mathbf{a}[x-y;A(x)]$$
$$= [z\cdot\nabla_x + z\cdot\partial_z]\mathrm{e}^{-z\cdot\nabla_x}1\Big|_{z=x-y} = 0\,. \quad (26)$$

In the literature, these constraints usually lead to the path ordered non-integratable phase factor $\mathbf{P}\mathrm{e}^{\mathrm{i}g\int_y^x \mathrm{d}z^{\mu}A_{\mu}(z)}$ [6], which depends on the path. Our result instead only relies on the end points $x, y$ and is independent of the path. Except the formal definition of $\mathbf{a}[x-y;A(x)]$, the explicit expression of $\mathbf{a}[x-y;A(x)]$ can be obtained with the help of the Baker–Hausdoff formula

$$\mathbf{a}[x-y;A(x)] = \left[\mathrm{e}^{-z\cdot\nabla_x}\mathrm{e}^{z\cdot\partial_x}1\right]_{z=x-y} \equiv \mathrm{e}^{C(x,z)}\Big|_{z=x-y}\,, \quad (27)$$

where $C(x,z)$ is defined by

$$\mathrm{e}^{C(x,z)} = \mathrm{e}^{-z\cdot\nabla_x}\mathrm{e}^{z\cdot\partial_x}$$
$$= \exp\Big[-z\cdot\nabla_x + z\cdot\partial_x + \frac{1}{2}[-z\cdot\nabla_x, z\cdot\partial_x]$$
$$+ \frac{1}{12}[-z\cdot\nabla_x,[-z\cdot\nabla_x,z\cdot\partial_x]]$$
$$-\frac{1}{12}[z\cdot\partial_x,[z\cdot\partial_x,-z\cdot\nabla_x]] + \cdots\Big]$$
$$= \exp\Big[\mathrm{i}gz\cdot A(x) + \frac{1}{2}[\mathrm{i}gz\cdot A(x), z\cdot\partial_x]$$
$$+ \frac{1}{12}[\mathrm{i}gz\cdot\nabla_x,[\mathrm{i}z\cdot A(x),z\cdot\partial_x]]$$
$$-\frac{1}{12}[z\cdot\partial_x,[z\cdot\partial_x,\mathrm{i}gz\cdot A(x)]] + \cdots\Big]\,. \quad (28)$$

Note that the $C(x,z)$ do not include the pure operator $\partial_x$, again, all the $\partial_x$ in the $C(x,z)$ are already acting on the gluon field $A_{\mu}(x)$ and the gluon field dependence in $C(x,z)$ is local at the space-time point $x$.

The gauge transformation covariance of $\mathbf{a}[x-y;A(x)]$ can be proved as follows: since under the gauge transformation $V(x)$, the $\nabla_x^{\mu}$ transform as $\nabla_x^{\mu} \to V(x)\nabla_x^{\mu}V^{\dagger}(x)$; the $\mathbf{a}[x-y;A(x)]$ then transform as

$$\mathbf{a}[x-y;A(x)] \to \left[\mathrm{e}^{-V(x)z\cdot\nabla_x V^{\dagger}(x)}1\right]_{z=x-y}$$
$$= V(x)\left[\mathrm{e}^{-z\cdot\nabla_x}V^{\dagger}(x)\right]_{z=x-y}$$
$$= V(x)\left[\mathrm{e}^{-z\cdot\nabla_x}\mathrm{e}^{z\cdot\partial_x}\mathrm{e}^{-z\cdot\partial_x}V^{\dagger}(x)\right]_{z=x-y}$$
$$= V(x)\mathrm{e}^{C(x,z)}\left[\mathrm{e}^{-z\cdot\partial_x}V^{\dagger}(x)\right]_{z=x-y}$$
$$= \left[V(x)\mathrm{e}^{C(x,z)}V^{\dagger}(x-z)\right]_{z=x-y}$$
$$= V(x)\mathbf{a}(x,y;A)V^{\dagger}(y)\,, \quad (29)$$

where we have used the property

$$\mathrm{e}^{-z\cdot\partial_x}V^{\dagger}(x)$$
$$= \left[1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}z_{\mu_1}\cdots z_{\mu_n}\partial_{x,\mu_1}\cdots\partial_{x,\mu_n}\right]V^{\dagger}(x)$$
$$= V^{\dagger}(x-z)\,.$$

Combining (17) and (29), we find our result that the quark propagator (14) is gauge covariant:

$$S(x, y; A) \to V(x)S(x, y; A)V^{\dagger}(y) \; . \tag{30}$$

Similarly since the Lorentz covariance for $\tilde{S}$ and $\mathbf{a}$ is explicit, our resulting propagator is explicitly Lorentz covariant.

In conclusion, we have factorized the quark propagator $S(x, y; A)$ by a generalized Fourier transformation of the momentum space quark propagator $\tilde{S}[x - y; A(x)]$ in the presence of a gluon field and a path independent "phase factor" $\mathbf{a}[x - y; A(x)]$. The two parts are all only dependent on the gluon field at the local space-time point $x$. The formalism is gauge and Lorentz covariant; it coincides with the exact propagator in the trivial case of a vanishing gluon field.

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